

PARRY'S TOPOLOGICAL TRANSITIVITY AND f -EXPANSIONS

E. ARTHUR ROBINSON, JR.

ABSTRACT. In his 1964 paper [21] on f -expansions, Parry studied piecewise-continuous, piecewise-monotonic maps F of the interval $[0, 1)$, and introduced a notion of topological transitivity different from any of the modern definitions. This notion, which we call *Parry topological transitivity*, (PTT) is that the *backward orbit* $O^-(x) = \{y : x = F^n y \text{ for some } n \geq 0\}$ of some $x \in [0, 1)$ is dense. We take *topological transitivity* (TT) to mean that some x has a dense *forward* orbit. Parry's application to f -expansions is that PTT implies the partition of $[0, 1)$ into the “fibers” of F is a generating partition (i.e., f -expansions are “valid”). We prove the same result for TT, and use this to show that for interval maps F , TT implies PTT. A separate proof is provided for continuous maps F of compact metric spaces. The converse is false.

1. INTRODUCTION

The concept of *topological transitivity* plays an important role in dynamical systems theory. Let $F : X \rightarrow X$ be a surjective map on a topological space X . The definition of topological transitivity (TT) that we will adopt in this paper is that for some $x \in X$ the *forward orbit* $O^+(x) = \{F^n x : n \geq 0\}$ is dense in X . Another definition, sometimes called *regional topological transitivity* (RTT), is that for any two non-empty open sets $U, V \subseteq X$, there exists $n > 0$ so that $U \cap F^n V \neq \emptyset$, or equivalently (see [1]), $F^{-n}U \cap V \neq \emptyset$. The equivalence of TT and RTT for continuous maps F of perfect compact metric spaces X is well known (see Proposition 1 below). Several papers (see for example [13] or [1]) discuss these, and other, definitions of topological transitivity for continuous maps F , and give conditions under which various definitions are equivalent.

However, one often wants to apply the concept of topological transitivity in situations with less ideal hypotheses. One benefit of the definition TT is that it makes sense even when F is not continuous. In this paper, we will mostly be interested piecewise monotonic, piecewise continuous maps F on the unit interval. In 1964, Parry [21] gave a different definition of topological transitivity in this situation, which we refer to here as *Parry topological transitivity* (PTT). It says that for some $x \in X$ the *backward orbit*

$$O^-(x) = \{F^{-n}(x) : n \geq 0\} = \{y : x = F^n(y) \text{ for some } n \geq 0\}$$

is dense.

As we will see, PTT generally does not imply TT, but in many situations, TT does imply PTT. It is not hard to obtain such results under “nice” hypotheses, like for subshifts (see Corollary 7) or for continuous maps F of perfect compact metric spaces (see Theorem 4). In this case, we show that TT implies that $O^-(x)$ is dense

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for a dense G_δ set of $x \in X$. Recently, it was shown in [17] that continuous TT maps $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfy PTT (in fact [17] proves more: if such an F is TT then $O^-(x)$ is dense for all but possibly two points $x \in \mathbb{R}$).

Our main goal in this paper is to understand the situation in the case studied by Parry [21], namely, for piecewise-continuous, piecewise-monotonic maps F of the interval. We call a surjective map $F : [0, 1) \rightarrow [0, 1)$ a *piecewise interval map* if there is a partition of (almost all of) $[0, 1)$ into a finite or countable ξ of disjoint intervals, indexed by a “digit set” \mathcal{D} , such that each $F|_{\Delta(d)}$ is continuous and strictly monotonic (see Section 3 for details). In his paper [21], Parry considered piecewise interval maps in the context of f -expansions, as defined by R enyi [22], Bissinger [3] and Everett [9] in the 1940’s and 1950’s. Unknown to these authors, the same idea had actually been studied earlier in 1929 by Kakeya [11].

The idea of f -expansions (the term is due to R enyi, [22]) is to use piecewise interval maps F to obtain what we call the F -representation $\mathbf{r}(x) \in \mathcal{D}^\mathbb{N}$ of $x \in [0, 1)$ by recording the sequence $\mathbf{r}(x) = .d_1 d_2 d_3 \dots$ of ξ -intervals visited by the F -iterates of x (see Section 3). The goal is to find conditions on F so that almost every x has a unique F -representation (“valid” in Parry’s terminology [21]). One also studies an algorithm (see Section 5) to recover x from $\mathbf{r}(x)$. In particular, under appropriate conditions the “ f -expansion” $f(d_1 + f(d_2 + f(d_3 + \dots)))$ converges to x , where $f : \mathbb{R} \rightarrow [0, 1]$ is a function satisfying $F(x) = f^{-1}(x) \bmod 1$.

There are, of course, two examples of F -representations and f -expansions that are especially well known. Binary representations/expansions of real numbers correspond to $F(x) = 2x \bmod 1$, with $f(x) = x/2$.¹ Continued fraction representations correspond to $F(x) = 1/x \bmod 1$. In each of these cases, F satisfies both TT and PTT.

In [21] Parry proved that PTT implies F -representations are valid (and also, with some additional hypotheses, that valid F -expansions implies PTT). In this paper, we prove a slightly strengthened version of Parry’s first result, as well as a “modern” version of Parry’s result, which says that TT implies F -representations are valid. One benefit is that TT is often very easy to verify. For example, (see Proposition 8), any F that is ergodic for an invariant measure μ equivalent to Lebesgue measure will satisfy TT. In the end, we show (Theorem 26) that TT implies PTT for piecewise interval maps F .

2. TOPOLOGICAL TRANSITIVITY

In this section we consider various notions of topological transitivity for continuous maps F . We begin with two standard results mentioned in the introduction, which we prove for the sake of completeness.

Proposition 1. *Suppose $F : X \rightarrow X$ is a continuous map on a compact metric space. If F satisfies RTT then the set $X_0 = \{x : O^+(x) \text{ is dense in } X\}$ contains a dense G_δ . In particular, RTT implies TT. If, in addition, X is perfect, then TT implies RTT.*

Proof. Assume $U \cap F^{-n}V \neq \emptyset$ for some $n \geq 0$. Then for any V open, $\cup_{n \geq 0} F^{-n}V$ is dense open, since it meets any open set U . Thus, whenever V_k is countable basis for X , the Baire Category Theorem implies $X_0 = \bigcap_{k \geq 0} \cup_{n \geq 0} F^{-n}V_k$ is dense G_δ . Clearly

¹Decimal representation and decimal expansions correspond to replacing the “base” 2 with base 10.

$x \in X_0$ implies that for any k , there exists $n \geq 0$ so that $F^n(x) \in V_k$. Thus $O^+(x)$ is dense.

Now suppose $O^+(x)$ is dense and let U and V be open. There exist $n, m \geq 0$ with $F^n x \in U$ and $F^m x \in V$. Since X is perfect, Lemma 2 (below) shows we may assume $m > n$. It follows that $F^n x \in U \cap F^{-m+n} V \neq \emptyset$. \square

Lemma 2. *Let X be a perfect (no isolated points) metric space (not necessarily compact), and suppose $F : X \rightarrow X$ is continuous. If $O^+(x)$ is dense, and $V \subseteq X$ is open, then $\{k \in \mathbb{N}_0 : F^k(x) \in V\}$ is infinite.*

Proof. Let $V_n = V \setminus \{F^k(x) : x = 0, 1, \dots, n-1\}$. Since X is perfect metric, V_n is nonempty and open, so $O^+(x) \cap V_n \neq \emptyset$. It follows that for any n there is $k \geq n$ so that $F^k(x) \in V$. \square

Proposition 3. *If $F : X \rightarrow X$ is a homeomorphism of a perfect compact metric space, then TT is equivalent to TTT. In fact, F and F^{-1} both satisfy TT, and there exists $X_0 \subseteq X$, dense G_δ so that $O^-(x)$ and $O^+(x)$ are both dense for $x \in X_0$.*

Proof. Clearly, if $O^+(x)$ is dense then $O(x)$ is dense for all x in a dense G_δ . On the other hand, if $O(x)$ is dense then either $O^+(x)$ is dense (for all x in a dense G_δ), and thus F is TT, or $O^-(x)$ is dense (for all x in a dense G_δ) and F^{-1} is TT. In the latter case, $U \cap F^{-n} V \neq \emptyset$ for some $n \geq 0$, which shows F is also TT. \square

2.1. The relation between TT and PTT. We begin with an example that shows Parry topological transitivity (PTT) does not imply topological transitivity (TT).

Example 1. Define a surjective map on a compact metric space by $F : [0, 1] \rightarrow [0, 1]$ by²

$$F(x) = \begin{cases} -2x + 1/2 & \text{if } x \in [0, 1/4), \\ 2x - 1/2 & \text{if } x \in [1/4, 3/4), \text{ and} \\ -2x + 5/2 & \text{if } x \in [3/4, 1]. \end{cases}$$

Note that $O^-(1/2)$ is dense, whereas $F^n(1/8, 3/8) \cap (5/8, 7/8) = \emptyset$ for all $n \geq 0$. Note also that just a single point has $O^-(x)$, and not a dense G_δ set of points.

In the other direction, we have the following:

Theorem 4. *Suppose F is a continuous map on a perfect compact metric space X that satisfies TT. Then F satisfies PTT, and moreover, the set $X_0 = \{x : O^-(x) \text{ is dense}\}$ contains a dense G_δ .*

Proof. Define

$$\tilde{X} = \{\tilde{x} = (x_1, x_2, x_3 \dots) \in X^{\mathbb{N}_0} : x_n = F(x_{n+1})\}.$$

This is a compact metric space with the topology induced by product topology, with metric

$$\tilde{d}(\tilde{x}, \tilde{y}) = \sum_{n \geq 1} d_X(x_n, y_n) / 2^n,$$

where d is the metric on x . Define $\tilde{F} : \tilde{X} \rightarrow \tilde{X}$ by

$$\tilde{F}(\tilde{x}) = (F(x_1), x_1, x_2, x_3 \dots),$$

²My thanks to Ethan Akin for suggesting this example.

and note that $\tilde{F}^{-1}(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$, so \tilde{F} is a homeomorphism. The map $\pi_k : \tilde{X} \rightarrow X$, defined $\pi_k(x_1, x_2, x_3, \dots) = x_k$, is surjective and open. It follows that \tilde{X} is perfect since X is perfect. The sets $\tilde{U}_k = \pi_k^{-1}(U)$, for $k = 1, 2, 3, \dots$ and $U \subseteq X$ open, form a sub-base for the topology on \tilde{X} . Given a countable base \mathcal{U} for X , let $\tilde{\mathcal{U}}$ consist of all nonempty sets of the form

$$(1) \quad \tilde{U} = \pi_1^{-1}(U_1) \cap \pi_2^{-1}(U_2) \cap \dots \cap \pi_\ell^{-1}(U_\ell) \subseteq \tilde{X},$$

for some $\ell \geq 1$ and $U_1, U_2, \dots, U_\ell \in \mathcal{U}$. Then $\tilde{\mathcal{U}}$ is a countable base for \tilde{X} .

For $\tilde{U} \in \tilde{\mathcal{U}}$, let

$$(2) \quad U = F^{-\ell+1}U_1 \cap F^{-\ell+2}U_2 \cap \dots \cap U_\ell \subseteq X,$$

and assume U is nonempty. Since F satisfies TT, the set

$$X_U = \bigcup_{n \geq 0} F^{-n}U = \{x \in X : O^+(x) \cap U \neq \emptyset\}$$

is dense open, so $X_0 = \bigcap_{U \in \tilde{\mathcal{U}}} X_U$ is dense G_δ . We **claim** that $O^+(\tilde{x})$ is dense in \tilde{X} for any $\tilde{x} \in \pi_1^{-1}(X_0)$, and thus \tilde{F} satisfies TT.

To prove the claim, fix $x \in X_0$ and let $\tilde{x} = (x, x_2, x_3, \dots) \in \pi_1^{-1}(X_0)$. Note that for any $k \geq 1$,

$$\tilde{F}^k(\tilde{x}) = (F^k(x), F^{k-1}(x), \dots, x, x_2, \dots).$$

Given $\tilde{U} \in \tilde{\mathcal{U}}$, let U be as in (2). Since $O^+(x)$ is dense and X is perfect, we can choose $n \geq \ell - 1$ so that $F^{n-\ell+1}x \in U$. Then

$$F^n(x) \in U_1, F^{n+1}(x) \in U_2, \dots, F^{n+\ell-1}(x) \in U_\ell,$$

and it follows from (1) that $\tilde{F}^n(\tilde{x}) \in \tilde{U}$. Since $\tilde{U} \in \tilde{\mathcal{U}}$ was arbitrary, $O^+(\tilde{x})$ is dense, proving the claim.

Now, since F satisfies TT and \tilde{F} is a homeomorphism of a perfect metric space \tilde{X} , it follows from Proposition 3 that \tilde{F} satisfies TTT. Thus $O^-(\tilde{x})$ dense for $\tilde{x} \in \tilde{X}_0 \subseteq \tilde{X}$, where \tilde{X}_0 contains a dense G_δ . Since π_1 is surjective and open, $X_0 = \pi_1(\tilde{X}_0)$ contains a dense G_δ , and for $\tilde{x} \in \tilde{X}_0$, $\pi_1(O^-(\tilde{x}))$ is dense in X . But $\pi_1(O^-(\tilde{x})) \subseteq O^-(\pi_1(\tilde{x})) = O^-(x)$, so F satisfies PTT. \square

For $F : X \rightarrow X$, we call a set $B^- \subseteq X$ a *backward orbit* of $x \in X$ if $x_1 = x$ and $B^- = \{x_1, x_2, x_3, \dots\}$ with $x_n = F(x_{n+1})$ for all $n \geq 1$. We say F satisfies *strong Parry topological transitivity* (STT) if there exists a dense backward orbit for some $x \in X$. Clearly STT implies PTT. The proof of Theorem 4 shows that under the same hypotheses, STT is equivalent to TT. Note that Example 1 does not satisfy STT although it does satisfy PTT.

2.2. PTT for symbolic dynamical systems. Consider the 1-sided full shift

$$\mathcal{D}^{\mathbb{N}} = \{\mathbf{d} = .d_1 d_2 d_3 \dots : d_j \in \mathcal{D}\},$$

where $2 \leq \#(\mathcal{D}) \leq \infty$, with left shift map $T(.d_1 d_2 d_3 \dots) = .d_2 d_3 \dots$. Consider also the 2-sided full shift $\mathcal{D}^{\mathbb{Z}}$ with left shift homeomorphism $\tilde{T}(\dots d_{-1}.d_0 d_1 d_2 \dots) = \dots d_{-1} d_0 .d_1 d_2 \dots$. We use the product topology in each case. If $\#(\mathcal{D}) < \infty$, then these are compact metric spaces, homeomorphic to the Cantor set, but in any case, they are uncountable, totally disconnected, Polish spaces.

Call a subset $X \subseteq \mathcal{D}^{\mathbb{N}}$ a *1-sided subshift* if it is closed and T invariant: $T(X) \subseteq X$. Similarly, call a subset $Y \subseteq \mathcal{D}^{\mathbb{Z}}$ a *2-sided subshift* if it is closed and \tilde{T} invariant: $\tilde{T}(Y) = Y$. The *language* \mathcal{L} of X (or \mathcal{L} of Y) is the set of all finite words $w =$

$w_0 w_1 \dots w_{\ell-1}$ (we say $|w| = \ell$) so that there exists $\mathbf{d} = .d_1 d_2 d_3 \dots \in X$ (or $\mathbf{e} = \dots d_{-1} d_0 . d_1 d_2 \dots \in Y$) and $k \in \mathbb{N}$ (or $k \in \mathbb{Z}$) with $w_0 w_1 \dots w_{\ell-1} = d_k d_{k+1} \dots d_{k+\ell-1}$. Given a 1-sided subshift X , we define its *natural extension* \tilde{X} to be the two sided subshift with the same language. A sub-basis for the topology on X is given by *cylinder sets*, which have the form $[w] = \{\mathbf{d} \in X : \mathbf{d}|_{[1,2,\dots,|w|]} = w\}$, for $w \in \mathcal{L}$. Similarly, a sub-basis for the topology on Y is given by *cylinder sets*, which have the form $[w] = \{\mathbf{e} \in Y : \mathbf{e}|_{[-\ell,-\ell+1,\dots,\ell-1,\ell]} = w\}$, where $w \in \mathcal{L}$ and $|w| = 2\ell + 1$. The following is an easy characterization of TT for $T: X \rightarrow X$ or $\tilde{T}: Y \rightarrow Y$.

Lemma 5. *The 1-sided (or 2-sided) shift, X (or Y), is topologically transitive (TT or TTT) if and only if its language \mathcal{L} satisfies*

$$(3) \quad \forall v, w \in \mathcal{L} \exists c \in \mathcal{L} \text{ so that } v c w \in \mathcal{L}.$$

Proof. Suppose $O^+(\mathbf{d})$ is dense in the 1-sided shift X . Given $v, w \in \mathcal{L}$, let $[v]$ and $[w]$ be the corresponding cylinder sets. We have that there exist $n, m \in \mathbb{N}_0$ so that $T^n(\mathbf{d}) \in [v]$ and $T^m(\mathbf{e}) \in [w]$, and by Lemma 2, we may assume $m \geq n + |v|$. We have $\mathbf{d}_{[n,\dots,n+|v|-1]} = v$ and $\mathbf{d}_{[m,\dots,m+|w|-1]} = w$. Since $m > n + |v| - 1$, $\mathbf{d}_{[n,\dots,m+|w|-1]} = v c w$ for some $c \in \mathcal{D}^{m-n-|v|+1} \cap \mathcal{L}$. Now suppose $O(\mathbf{e})$ is dense in Y . We are done if $O^+(\mathbf{e})$ is dense, so assume that $O^-(\mathbf{e})$ is dense. Then there are $n, m \in \mathbb{N}$ so that $T^{-n}(\mathbf{e}) \in [v]$ and $T^{-m}(\mathbf{e}) \in [w]$, and $-n + |v| \leq -m$. Thus $\mathbf{e}_{[-n,\dots,-m+|w|-1]} = v c w$ for some $c \in \mathcal{L}$.

Conversely, suppose \mathcal{L} satisfies (3). Enumerate $\mathcal{L} = \{w_1, w_2, w_3, \dots\}$, and by induction, choose a sequence $c_1, c_2, c_3, \dots \in \mathcal{L}$ so that $w_1 c_1 w_2 c_2 \dots w_{n-1} c_{n-1} c_n \in \mathcal{L}$ for all n . Then $\mathbf{d} = w_1 c_1 w_2 c_2 w_3 c_3 w_4 \dots \in X$ and $O^+(\mathbf{d})$ is dense. In a similar way, if \mathcal{L} satisfies (3) for a two sided shift Y , then there exists $\mathbf{e} = \dots w_3 b_2 w_2 b_1 w_1 c_1 w_2 \dots$ with $O(\mathbf{e})$ dense. \square

Corollary 6. *The 2-sided natural extension \tilde{X} satisfies TTT (equivalently, TT) if and only if the corresponding 1-sided shift X satisfies TT.*

Corollary 7. *If a 1-sided shift X satisfies TT then it satisfies PTT.*

Proof. The natural extension \tilde{X} of X is TT, and there exists $\tilde{\mathbf{d}} \in \tilde{X}$ so that $O^-(\tilde{\mathbf{d}})$ is dense. The 1-sided factor map $\pi_+ : \tilde{X} \rightarrow X$, defined $\pi_+(\dots d_{-1} d_0 . d_1 d_2 \dots) = .d_1 d_2 \dots$, is open and satisfies $\pi_+(\tilde{T}(\tilde{\mathbf{d}})) = T(\pi_+(\tilde{\mathbf{d}}))$. Then for $\mathbf{d} = \pi_+(\tilde{\mathbf{d}})$, we have that $O^-(\mathbf{d}) \subseteq \pi_+(O^-(\tilde{\mathbf{d}}))$ is dense in X . \square

Note that when $\#(\mathcal{D}) < \infty$, Corollary 6 and Corollary 7 follow from Theorem 4. However, we will also be interested in the case $\#(\mathcal{D}) = \infty$.

Example 2. Let $X \subseteq \{1, 2, \bar{1}, \bar{2}\}^{\mathbb{N}}$ be the subshift defined by forbidding the words $\mathcal{F} = \{\bar{k}\ell : k, \ell \in \{1, 2\}\}$. Here we have two 1-sided 2-shifts: “unbarred” $\{1, 2\}^{\mathbb{N}}$ and “barred” $\{\bar{1}, \bar{2}\}^{\mathbb{N}}$, with the possibility of “barring” the tail of a point $\mathbf{d} \in \{1, 2\}^{\mathbb{N}}$. Any point $\mathbf{d} \in X$ has $T^n(\mathbf{d}) \in \{1, 2\}^{\mathbb{N}} \cup \{\bar{1}, \bar{2}\}^{\mathbb{N}}$ for n sufficiently large, so $O^+(\mathbf{d})$ is never dense. However, any $\bar{\mathbf{d}} \in \{\bar{1}, \bar{2}\}^{\mathbb{N}}$ has $O^-(\bar{\mathbf{d}})$ dense.

3. PIECEWISE INTERVAL MAPS

Let λ denote Lebesgue measure on $[0, 1)$. An *interval partition* is a finite or countable indexed collection $\xi = \{\Delta(d) \subseteq [0, 1) : d \in \mathcal{D}\}$ of $2 \leq \#(\xi) \leq \infty$ disjoint intervals, with $\lambda(D) = 1$, where $D = \bigcup_{d \in \mathcal{D}} \Delta(d)$. The intervals $\Delta(d)$, which have endpoints $a_d < b_d$, may be open, closed, half open $(a_d, b_d]$ or half-closed $[a_d, b_d)$. Let

$\Delta^\circ(d) = (a_d, b_d)$ and note that $\cup_{d \in \mathcal{D}} \Delta^\circ(d)$ is always dense and open. We generally refer to elements of the index set \mathcal{D} as *digits*.

A *piecewise interval map* (PIM) F on $[0, 1)$ is an interval partition ξ , together with a map $F : D \rightarrow [0, 1)$ such that

- (1) each $F|_{\Delta(d)}$ is continuous and strictly monotonic,
- (2) $\lambda(B^c) = 0$ where $B = \{x : F^n x \in D \text{ for all } n \geq 0\}$,
- (3) for all $d, d' \in \mathcal{D}$ (including $d = d'$) and $n \geq 0$, $\Delta(d) \cap T^n \Delta(d')$ is either an interval or empty (i.e., it does not consist of a single point). Equivalently, $\Delta^\circ(d) \cap T^n \Delta^\circ(d') \neq \emptyset$.

We often also assume that F is surjective (and this is clearly necessary for F to satisfy TT), although we do not require this. For a PIM F , we say $F|_\Delta$ is *type A* if it is increasing and *type B* if it is decreasing. We say F is type A (or type B) if every $F|_\Delta$ is type A (or type B). Otherwise, F is called *mixed type*. We say F is *full* on $\Delta \in \xi$ if $F(\Delta) = [0, 1)$. Condition (4) can always be achieved by taking each $\Delta \in \xi$ to be an open interval. The process of removing some endpoints from ξ to make F satisfy (4) only changes D on a countable set. However, in certain examples, it is natural to keep the endpoints (see the examples below).

Since each $F|_\Delta$ is strictly monotonic, condition (3) is automatic if D^c is countable. In particular, (3) always holds if ξ is finite. Condition (3) also holds if $\lambda(\{x : F'(x) = 0\}) = 0$, since this is equivalent to F being *nonsingular* in the sense that $\lambda(F^{-1}E) = 0$ for each $E \subseteq [0, 1)$ with $\lambda(E) = 0$.

In many cases (see e.g., [6], [5]) one can show more. We say a measure μ on $[0, 1)$ is an *absolutely continuous* F -invariant measure, *equivalent* to Lebesgue measure (ACIM), if there exists an integrable function with $\rho(x) > 0$ for λ a.e. $x \in [0, 1)$ so that μ , defined by $\mu(E) = \int_E \rho(x) dx$, is F -invariant, namely, $\mu(F^{-1}E) = \mu(E)$. In particular, the existence of an ACIM implies that F is nonsingular. Quite often one can also show that this ACIM μ is an *ergodic* measure for F (see the examples below). The following is a routine application of the Birkhoff ergodic theorem.

Proposition 8. *If a PIM F has an ergodic ACIM then F satisfies TT.*

3.1. F -representations. Recall that $D = \cup_{d \in \mathcal{D}} \Delta(d) \subseteq [0, 1)$. By abuse of notation, we also denote by ξ the map $\xi : D \rightarrow \mathcal{D}$ with $\xi(x) = d$ for $x \in \Delta(d)$. Given a PIM F , we define the F -representation of $x \in B$ to be the sequence

$$\mathbf{r}(x) = .d_1 d_2 d_3 d_4 \cdots \in \mathcal{D}^\mathbb{N},$$

where $d_n = \xi(F^{n-1}x)$ for $n \in \mathbb{N}$. In ergodic theory, $\mathbf{r}(x)$ is called the (F, ξ) -name of x . We say that F -representations are *valid* if the map \mathbf{r} is injective for λ a.e. $x \in B$.

Parry observed in his paper [21] that the previous conditions for validity (i.e., in [3], [9], [22]) were sufficient conditions, and were “metric” in nature. Probably the nicest result of this type is Kakeya’s Theorem [11], which essentially says that F -representations are valid for PIMs F of type A or B, provided $|F'(x)| > 1$ almost everywhere. Parry observed that one ought to expect the necessary and sufficient conditions for validity to be *dynamical* in nature. He went on to prove that F -representations are valid if F satisfies what we have called Parry topological transitivity.

3.2. Examples.

Example 3 (β -representations). Consider the type A maps $F : [0, 1] \rightarrow [0, 1]$ defined by $F(x) = \beta x \bmod 1$, for $\beta > 1$. Here $\xi(x) = \lfloor \beta x \rfloor$ with $\mathcal{D} = \{0, 1, \dots, \beta - 1\}$ for $\beta \in \mathbb{N}$,

and $\mathcal{D} = \{0, 1, \dots, \lfloor \beta \rfloor\}$ for $\beta \notin \mathbb{N}$. The β -representations were introduced in [22], who showed that every β -transformation F has an ergodic ACIM (so satisfy TT). An explicit formula for the density $\rho(x)$ was given by Parry [20]). Parry (see [21]) studied the more general α - β -transformation, $F(x) = \alpha + \beta x \bmod 1$, which he showed are not necessarily ergodic (or topologically transitive).

Example 4 (Generalized Gauss transformations). For real numbers $r \geq 1$, define the type 2 map $F(x) = r/x \bmod 1$ with $\xi(x) = \lfloor r/x \rfloor$. The case $r = 1$, known as the *Gauss transformation*, has an ergodic ACIM with $\rho(x) = (\log(2)(x+1))^{-1}$. The existence of an ergodic ACIM for $r > 1$ is proved in [15] (an explicit formula for each $r \in \mathbb{N}$ is given in [7]). Thus each such F satisfies TT. The corresponding F -representations are (generalized) continued fraction coefficients.

Example 5 (Quadratic maps). For $s \approx 0.8$, $s < r \leq 1$, consider $F : [0, 1] \rightarrow [0, 1]$ by

$$F(x) = -4r \left((1 - r - 4r^2 + 4r^3) - (1 - 8r^2 + 8r^3)x + r(1 - 2r)^2 x^2 \right),$$

with $\xi(x) = 0$ if $x < (1 + 2r - 4r^2)/(2r - 4r^2)$ and $\xi(x) = 1$ otherwise (this is the map $q(x) = 4rx(1 - x)$, restricted to the interval $[q(r), r]$, then renormalized). These maps are commonly studied in chaos theory (see [8]). There is a set r of positive Lebesgue measure with an ergodic ACIM and hence TT. It is known that there is a set of values for r of positive Lebesgue measure so that F has an ergodic ACIM and hence is TT. Closely related to both the quadratic maps and β -transformations are the *tent maps* defined for $1 < \tau \leq 2$ by $P(x) = \tau x \bmod 1$, where we define $y \bmod 1 = y \bmod 1$ if $\lfloor y \rfloor$ is even, and $1 - (y \bmod 1)$ if $\lfloor y \rfloor$ is odd. For all τ sufficiently large, F has an ergodic ACIM and hence is TT (see [10]).

Example 6 (The Cantor map). This map F is defined to be linear, increasing, and full on each intervals ξ in the complement K^c of the Cantor set K . The intervals in ξ are naturally indexed by $\mathcal{D} = \mathbb{Z}[1/2] \cap (0, 1)$, the dyadic rationals in $(0, 1)$. Note that in this example $D = K^c$ is measure zero but uncountable. More generally, ξ can be replaced by any interval partition. Such maps are called *generalized Lüroth transformations* in [6] in the case of ξ finite. All such maps are TT and ergodic for Lebesgue measure.

Example 7 (Generalized Egyptian fractions). Define $F(x) = x - 1/\lfloor 1/x \rfloor$ and $\xi(x) = \lfloor 1/x \rfloor$. Note that $O^-(0)$ is dense, so F satisfies PTT, whereas $F^n(x) \searrow 0$ for all x , so F does not satisfy TT. Note also that $O^-(x)$ is dense only for $x = 0$, and not for a dense G_δ set of x . Here, B is the set of irrationals, and $x = 1/d_1 + 1/d_2 + 1/d_3 \dots$ is the infinite *greedy Egyptian fraction* expansion of an irrational x . More generally, for a strictly increasing sequence $\mathbf{a} = (a_1, a_2, a_3, \dots)$ of positive integers, $a_1 > 1$, such that $1 \leq \sum 1/a_n \leq \infty$ (e.g. the primes). Let $\lfloor y \rfloor_{\mathbf{a}} = a_n$ where $a_{n-1} < y \leq a_n$, and $F(x) = x - 1/\lfloor 1/x \rfloor_{\mathbf{a}}$. The case $a_n = 2^n$ gives binary expansions.

Example 8 (Interval exchange transformations). Let ξ be an interval partition, and let ξ' be a “permutation” of ξ . In particular, suppose there is a bijection $\varphi : \xi \rightarrow \xi'$ such that each $\Delta \in \xi$ there is $r(\Delta) \in (-1, 1)$ so that $\varphi(\Delta) = \Delta + r(\Delta)$. Define $F(x) = x + r(\Delta)$ for $x \in \Delta$ (see [12], [19]). Interval exchanges preserve Lebesgue measure. Various conditions for ergodicity and TT are known (see [12], [24], [14]). Included here are the circle rotations F , which can be realized as 2-exchanges $\xi = \{[0, \alpha], [\alpha, 1]\}$ (labeled 0 and 1), with TT and ergodicity if and only if $\alpha \notin \mathbb{Q}$. The resulting F -representations are Sturmian sequences. Similarly, the

von Neumann adding machine transformation F is an exchange of the partition ξ into intervals of lengths $1/2^n$, in order of decreasing length, ξ' the partition into the same intervals, but in order of increasing lengths. This is TT and ergodic. Up to metric isomorphism, any ergodic measure preserving transformation F can be realized as a (usually infinite) interval exchange (see [2]). It should be noted that interval exchange transformations F differ from the other examples discussed here because they are invertible. Orientation reversing interval exchange transformations were studied in [18], but they rarely satisfy TT.

4. PARRY'S THEOREM

In this section we state and prove our main results about topological transitivity and valid F -expansions for piecewise interval maps F . The first result is Parry's theorem [21]. Our contribution is to extend the proof to the mixed type case.

Theorem 9 (Parry [21]). *Suppose F is a PIM (type A, type B or mixed type). If F satisfies PTT, then F -representations are valid.*

Parry also proved the following partial converse, which we prove below for convenience.

Proposition 10 (Parry, [21]). *Let F be a PIM so that $F^{-1}(0)$ includes all the endpoints of ξ except possibly 0 or 1. If F -representations are valid then F satisfies PTT.*

We also prove this below. Next, we state our “modern” version of Parry's Theorem.

Theorem 11. *Suppose F is a PIM (type A, type B or mixed type). If F satisfies TT then F -representations are valid.*

4.1. Some preliminaries. Let F be a PIM. An interval $I \subseteq [0, 1)$ is called a *homterval* if $F^n|_I$ is continuous and strictly monotonic for each $n \geq 1$. In particular, F is a homeomorphism between I and each $F^n(I)$. There are two special kinds of homtervals. A homterval I is called a *wandering interval* if $F^n(I) \cap F^m(I) = \emptyset$ for all $m > n \geq 0$. A homterval I is called *absorbing interval* with period $p \geq 1$ if $I, F(I), \dots, F^{p-1}(I)$ pairwise disjoint, and $F^p(I) \subseteq I$. Here we have that $F_I^p : I \rightarrow J$, for $J = F^p(I)$ is a homeomorphism. The following is a basic result of 1-dimensional dynamics (see [23]).

Lemma 12. *If J is a homterval, then either J is a wandering interval or $J \subseteq I$ for an absorbing interval I with some order p .*

Proof. Suppose J is a homterval that is not a wandering interval. Then there exist $n \geq 0$ and $p \geq 1$ so that $F^n J \cap F^{n+p} J \neq \emptyset$, and the interval $F^n J \cup F^{n+p} J$ is a homterval. Repeatedly applying F^p gives $F^{n+\ell p} J \cap F^{n+(\ell+1)p} J \neq \emptyset$ for each $\ell \geq 0$. It follows that $I = \bigcup_{\ell=0}^{\infty} F^{n+\ell p} J$ is a homterval with $J \subseteq I$. Moreover, $I, F(I), \dots, F^{p-1}(I)$ are pairwise disjoint and $F^p|_I$ is a homeomorphism from I onto a subinterval: $F^p(I) \subseteq I$. \square

Lemma 13. *If a PIM F satisfies TT, then there can be no homtervals.*

Proof. Suppose to the contrary that J is a homterval and $O^+(x)$ is dense. If J is a wandering interval, then $O^+(x)$ can meet J at most once. This contradicts the density of $O^+(x)$. By Lemma 12, the only other possibility is that there is period $p \geq 1$ absorbing interval I with $J \subseteq I$. Since $O^+(x)$ is dense, $F^n(x) \in O^+(x) \cap I$ for

some $n \geq 0$. We can assume without loss of generality that $n = 1$ so that $x \in I$. We claim this implies that $O^+(x) \cap I$ is not dense, which is a contradiction.

To prove the claim we note that $F^n(x) \in I$ only if $n = kp$ for some k (since I has period p), so without loss of generality we may assume $p = 1$, and assume F maps I homeomorphically onto $F(I)$. Let $x \in I$ and consider $O^+(x)$. One possibility is that $F(x) = x$, in which case $O^+(x) = \{x\}$ which is not dense in I . Now we divide into two cases: either $F|_I$ is strictly increasing or $F|_I$ is strictly decreasing. In the increasing case, suppose $F(x) \neq x$, and assume without loss of generality that $F(x) > x$. Then there is fixed point $F(y) = y \in I$ so that $F^n(x)$ is increasing and $F^n(x) \nearrow y$. Again $O^+(x)$ is not dense. In the decreasing case, we replace $F|_I$ with $(F^2)|_I$, which is increasing. □

Note that a period p absorbing interval I always contains at least one point y of period p . That point satisfies $O^+(y) = \{x, F(y), \dots, F^{p-1}(y)\}$, with $F^p(y) = y$. The iterates of any non-periodic point $x \in I$ limit onto some finite $O^+(y)$ (size p), as in the proof. This situation is described in [23] by saying that F has a *periodic attractor* of period p .

Lemma 14. *If a PIM F satisfies PTT then there can be no absorbing interval.*

This observation is essentially due to Parry [21].

Proof. Let I be an absorbing interval of period p . First, as in the proof of Lemma 13, we assume without loss of generality that $p = 1$, so that $F|_I : I \rightarrow J$, $J = F(I) \subseteq I$, is a homeomorphism. If $x \notin J$ then $F^{-1}(x) \cap I = \emptyset$, so $O^-(x)$ cannot be dense. Thus we assume $x \in J$, and show that $O^-(x)$ is not dense in J .

Consider the homeomorphism $(F|_I)^{-1} : J \rightarrow I$. We assume without loss of generality that $(F|_I)^{-1}$ is increasing (otherwise replace $F|_I$ with $(F|_I)^2$ and J with $(F|_I)^2(I)$). If there is an $n > 0$ so that $(F|_I)^{-n}(x) \notin J$ then $O^-(x) \cap J$ is finite. Thus we assume $(F|_I)^{-n}(x) \in J$ for all $n \geq 0$. One possibility is that $(F|_I)(x) = x$, but this implies $O^-(x)$ is not dense in J . Thus assume that $(F|_I)(x) > x$ (the case $(F|_I)(x) < x$ is analogous). This implies that $(F|_I)^{-n}(x)$ is a bounded increasing sequence (the graph of $(F|_I)^{-1}$ is above the diagonal on a neighborhood of x). In particular, $(F|_I)^{-n}(x)$ is not dense. □

Next we study iterations of the the partition ξ . For $d_1 d_2 \dots d_n \in \mathcal{D}^n$, let

$$\Delta(d_1 d_2 \dots d_n) = \{x : \mathbf{r}(x)_{[1, \dots, n]} = d_1 d_2 \dots d_n\}.$$

Equivalently,

$$\begin{aligned} \Delta(d_1 d_2 \dots d_n) &= \Delta(d_1) \cap F^{-1} \Delta(d_2) \cap \dots \cap F^{-n+1} \Delta(d_n) \\ (4) \quad &= \Delta(d_1) \cap F^{-1} \Delta(d_2 d_3 \dots d_n) \\ &= \Delta(d_1 d_2 \dots d_{n-1}) \cap F^{-n+1} \Delta(d_n). \end{aligned}$$

By our assumption (3) on F , the set $\Delta(d_1 d_2 \dots d_n)$ is either empty or a nontrivial interval. In the latter case, we call it a *fundamental interval* of order n (or a *cylinder*). Let $\xi^{(n)}$ be the interval partition into fundamental intervals of order n , and define $\|\xi^{(n)}\| = \sup\{|\Delta| : \Delta \in \xi^{(n)}\}$, where $|\Delta|$ denotes the length of Δ . It is clear that \mathbf{r} is injective if and only if $\|\xi^{(n)}\| \rightarrow 0$. In ergodic theory, one usually

writes

$$\xi^{(n)} = \bigvee_{k=1}^n F^{-k+1}\xi.$$

If $\|\xi^{(n)}\| \rightarrow 0$ then ξ is called a *generating partition* for F .

Proof of Proposition 10. Denote the endpoints of ξ by $|\xi|$. By the hypotheses $|\xi| = F^{-1}(0) \cup \{0, 1\}$, and similarly $|\xi^{(n)}| = \bigcup_{k=0}^{n-1} F^{-k}(0) \cup \{0, 1\}$. Since F -representations are valid, $\|\xi^{(n)}\| \rightarrow 0$, which implies $O^-(0) \cup \{0, 1\} = \bigcup_{n \geq 1} |\xi^{(n)}|$ is dense. It follows that F is PTT. \square

For $x \in B$, let $\Delta^n(x)$ be the interval in $\xi^{(n)}$ that contains x . Thus, $\|\xi^{(n)}\| \not\rightarrow 0$ if and only if there exists an x so that $|\Delta^n(x)| \not\rightarrow 0$. Note that $\Delta^{n+1}(x) \subseteq \Delta^n(x)$. Define

$$\Delta(x) = \bigcap_{n \in \mathbb{N}} \Delta^n(x)$$

Either $\Delta(x)$ is a (nontrivial) interval or $\Delta(x) = \{x\}$, with the former if and only if $|\Delta^n(x)| \not\rightarrow 0$, (i.e., if and only if F -representations are not valid).

All $y \in \Delta(x)$ satisfy $\mathbf{r}(y) = \mathbf{r}(x)$ and $\Delta(y) = \Delta(x)$. When $\Delta(x)$ is a nontrivial interval, each map $(F^n)|_{\Delta(x)}$, for $n \in \mathbb{N}$, is continuous and strictly monotonic (i.e., a homeomorphism onto its range). In particular, such an interval $\Delta^n(x)$ is a homterval. We summarize these last few paragraphs in a lemma.

Lemma 15. *If F -representations are not valid then there exists $x \in B$ so that $\Delta(x)$ is a homterval.*

Proof of Theorem 11. Suppose F -representations are not valid. By Lemma 15 there is a homterval $\Delta(x)$, and by Lemma 13, F cannot be TT. \square

4.2. Flip lexicographic order. Let $\mathcal{A} = \{d \in \mathcal{D} : F|_{\Delta(d)} \text{ is increasing}\}$ and $\mathcal{B} = \{d \in \mathcal{D} : F|_{\Delta(d)} \text{ is decreasing}\}$, so that $\mathcal{D} = \mathcal{A} \cup \mathcal{B}$ is a disjoint union. Note that $\mathcal{D} = \mathcal{A}$ if F is type A, and $\mathcal{D} = \mathcal{B}$ if F is type B. For two intervals $\Delta, \Delta' \in \xi$ say $\Delta < \Delta'$ if $x < x'$ for all $x \in \Delta$, $x' \in \Delta'$. This induces an order on \mathcal{D} by $d < d'$ if $\Delta(d) < \Delta(d')$. This order, in turn, leads to the following order on $\mathcal{D}^{\mathbb{N}}$ called *flip lexicographic order*.

Definition 16. *Suppose $\mathcal{D} \subseteq \mathbb{Z}$. Given $\mathbf{d} = .d_1 d_2 d_3 \dots \in \mathcal{D}^{\mathbb{N}}$, $\mathbf{e} = .e_1 e_2 e_3 \dots \in \mathcal{D}^{\mathbb{N}}$, with $\mathbf{d} \neq \mathbf{e}$, let $n = \min\{j \geq 1 : d_j \neq e_j\}$. Let $p = 0$ if $n = 1$ and otherwise $p = \#\{j = 1, \dots, n-1 : d_j = e_j \in \mathcal{B}\}$. Define $\mathbf{d} < \mathbf{e}$ if $d_n < e_n$ and p is even, or if $d_n > e_n$ and p is odd. Otherwise, define $\mathbf{e} < \mathbf{d}$. We will write $\mathbf{d} \leq \mathbf{e}$ if $\mathbf{d} < \mathbf{e}$ or $\mathbf{d} = \mathbf{e}$.*

If F is type A, this is lexicographic order, and if F is Type B, it is alternating lexicographic order. Parry's proof [21] of Theorem 9 assumes one of these two cases. Flip lexicographic order appears in [16].

Lemma 17. *If $x < y$ then $\mathbf{r}(x) \leq \mathbf{r}(y)$. Conversely, if $\mathbf{r}(x) < \mathbf{r}(y)$ then $x < y$. In particular, if $\mathbf{r}(x) \neq \mathbf{r}(y)$ then $x \neq y$.*

Proof. Let $x < y$ and $\mathbf{d} = \mathbf{r}(x)$ and $\mathbf{e} = \mathbf{r}(y)$. One possibility is that $y \in \Delta(x)$, so $\Delta(x) = \Delta(y)$, in which case, $\mathbf{d} = \mathbf{e}$. Otherwise there is a smallest $n \geq 1$ so that $\Delta^n(x) \neq \Delta^n(y)$. If $n = 1$, then $\Delta^1(x) = \Delta(d_1) < \Delta(e_1) = \Delta^1(y)$, so $d_1 < e_1$. Since $p = 0$, this implies $\mathbf{r}(x) < \mathbf{r}(y)$. If $n > 1$ then $x, y \in \Delta(d_1 d_2 \dots d_{n-1})$ and $p \leq n-1$. If p is even, $F^{n-1}|_{\Delta(d_1 d_2 \dots d_{n-1})}$ is increasing, and since $x < y$, $F^{n-1}(x) < F^{n-1}(y)$. We then have $\Delta(d_n) < \Delta(e_n)$ so that $d_n < e_n$. This implies $\mathbf{d} < \mathbf{e}$ since p is even. If, on

the other hand, p is odd, then $F^{n-1}|_{\Delta(d_0 d_1 \dots d_{n-1})}$ is decreasing, and $x < y$ implies $F^{n-1}(y) < F^{n-1}(x)$, which implies $\Delta(e_n) < \Delta(d_n)$ and $e_n < d_n$. Since p is odd, this still implies $\mathbf{d} < \mathbf{e}$.

Conversely, suppose $\mathbf{r}(x) < \mathbf{r}(y)$. If $d_1 < e_1$ then $\Delta(d_1) < \Delta(e_1)$ and $x < y$. Now suppose $x, y \in \Delta(d_1 d_2 \dots d_{n-1})$, but $d_n \neq e_n$. Since $\mathbf{x} < \mathbf{y}$, we have $d_n < e_n$ if p is even and $e_n < d_n$ if p is odd. In the first case we have $F^n(x) < F^n(y)$ and in the second, $F^n(y) < F^n(x)$ (because $F^n(x) \in \Delta(x_n)$, and likewise for y). Note that $F^n|_{\Delta(x_0, x_1, \dots, x_{n-1})}$ is continuous, and either increasing or decreasing, depending on whether p is even or odd. In both cases, this implies $x < y$. \square

Lemma 18. *Let F satisfy PTT, and let x be such that $O^-(x)$ is dense in $[0, 1)$. Then $\Delta(x) = \{x\}$.*

Proof. If $\Delta(x) \neq \{x\}$, then by Lemma 15, $\Delta(x)$ is a homterval. Since F satisfies PTT, Lemma 14 implies $\Delta(x)$ cannot be an absorbing interval, so by Lemma 12, $\Delta(x)$ must be a wandering interval. We show this is impossible.

Suppose $F^n(\Delta(x)) \cap F^m(\Delta(x)) = \emptyset$ for all $m > n \geq 0$. This is equivalent to $F^{-m}(\Delta(x)) \cap F^{-n}(\Delta(x)) = \emptyset$ for all $n > m \geq 0$. Now $F^{-n}(x) \subseteq F^{-n}(\Delta(x))$ for all n , but this containment is never dense. It follows that $O^-(x) = \cup_{n \geq 0} F^{-n}(x)$ cannot be dense in $[0, 1)$.

Thus $\Delta(x) = \{x\}$ as claimed. \square

Proof of Theorem 9. First note that $\Delta(z) = \{z\}$ whenever $Fz = y$ and $\Delta(y) = \{y\}$. Thus for any x with $O^-(x)$ dense, $z \in O^-(x)$ implies $\Delta(z) = \{z\}$.

Let $u < v$ and take $y, z \in O^-(x)$ so that $u < y < z < v$. By Lemma 17, $\mathbf{r}(u) \leq \mathbf{r}(y) < \mathbf{r}(z) \leq \mathbf{r}(v)$, so that $\mathbf{r}(u) < \mathbf{r}(v)$. Then by Lemma 17 again, $\mathbf{r}(u) \neq \mathbf{r}(v)$. \square

5. f -EXPANSIONS AND A GENERALIZATION

Given a PIM F , define the F -shift

$$X = \overline{\{\mathbf{r}(x) : x \in B\}} \subseteq \mathcal{D}^{\mathbb{N}},$$

with the left shift map T . Indeed, this is a 1-sided shift since $T(\mathbf{r}(x)) = \mathbf{r}(F(x))$. Let \tilde{X} , with \tilde{T} , be the 2-sided natural extension of X , and let \mathcal{L} be the language common to both shifts.

Lemma 19. *A word $d_1 d_2 \dots d_n \in \mathcal{L}$ if and only if $\Delta(d_1 d_2 \dots d_n)$ is an interval, or equivalently, $\Delta^\circ(d_1 d_2 \dots d_n) \neq \emptyset$*

Proof. Note that $w \in \mathcal{L}$ if and only if $w = \mathbf{r}(x)_{[1, 2, \dots, n]} = .d_1 d_2 \dots d_n$ for some $x \in B$. Then by (3) and (4), $\Delta(d_1, d_2, \dots, d_n)$ is an interval. Conversely, suppose $\Delta(d_1, d_2, \dots, d_n)$ is an interval. Let $x \in B \cap \Delta(d_1, d_2, \dots, d_n)$. Then $.d_1 d_2 \dots d_n = \mathbf{r}(x)_{[1, 2, \dots, n]} \in \mathcal{L}$ since $\mathbf{r}(x) \in X$. \square

For $w = d_1 d_2 \dots d_n \in \mathcal{L}$, let $\overline{\Delta}(d_1, d_2, \dots, d_n) = [a_n, b_n]$, so $\Delta^\circ(d_1, d_2, \dots, d_n) = (a_n, b_n)$. Note that $\overline{\Delta}(d_1, d_2, \dots, d_n) \subseteq \overline{\Delta}(d_1, d_2, \dots, d_{n-1})$. Thus if F representations are valid, $|\overline{\Delta}(d_1, d_2, \dots, d_n)| \rightarrow 0$ as $n \rightarrow \infty$ for any $\mathbf{d} = .d_1 d_2 d_3 \dots \in X$. Then $\{x\} = \cap_n \overline{\Delta}(d_1, d_2, \dots, d_n)$ and we define $E(\mathbf{d}) = x$. If $\mathbf{d} = \mathbf{r}(x)$ for $x_0 \in B$, then $x \in \Delta(d_1, d_2, \dots, d_n)$ for all n , so in this case, $E(\mathbf{r}(x)) = x$. We summarize.

Proposition 20. *Suppose F -representations are valid. Then for every $\mathbf{d} = .d_1d_2d_3\cdots \in X$ there exists a unique $x := E(\mathbf{d}) \in [0, 1]$ so that $\{x\} = \bigcap_n \overline{\Delta}(d_1, d_2, \dots, d_n)$. In particular, then $E(\mathbf{d}) = \lim_n a_n = \lim_n b_n$. If $x \in B$ and $\mathbf{d} = \mathbf{r}(x)$ then $E(\mathbf{d}) = x$.*

Lemma 21. *If $O^+(x)$ is dense and $\Delta^\circ(d_1, d_2, \dots, d_n) \neq \emptyset$, then $\{N : F^N(x) \in \Delta^\circ(d_1, d_2, \dots, d_n)\}$ is infinite.*

Proof. Since $O^+(x)$ is dense and $\Delta^\circ(d_1d_2\cdots d_n)$ is nonempty and open, there exists smallest $k_1 \geq 0$ so that $F^{k_1}(x) \in \Delta^\circ(d_1d_2\cdots d_n)$. We will show there exists $k_2 > k_1$ so that $F^{k_2}(x) \in \Delta^\circ(d_1d_2\cdots d_n)$.

We know that $\mathbf{r}(F^{k_1}(x))_{[1,2,\dots,n]} = .d_1d_2\cdots d_n$ and $\Delta^\circ(d_1d_2\cdots d_nd_{n+1}\cdots d_m) \subseteq \Delta^m(F^{k_1}(x))$ for all $m > n$. Since $O^+(x)$ is dense, F satisfies TT, and thus Theorem 11 implies F -representations are valid. This implies that $|\Delta^m(F^{k_1}(x))| \rightarrow 0$ as $m \rightarrow \infty$. It follows that for some $m > n$, which we choose as small as possible, $\Delta^\circ(d_1d_2\cdots d_m)$ is properly contained in $\Delta^\circ(d_1d_2\cdots d_n)$, and $F^{k_1}(x) \in \Delta^\circ(d_1d_2\cdots d_m)$. Then there exists $e_m \neq d_m$ so that $\Delta^\circ(d_1d_2\cdots d_{m-1}e_m) \neq \emptyset$, $\Delta^\circ(d_1d_2\cdots d_{m-1}e_m) \subseteq \Delta^\circ(d_1d_2\cdots d_m)$ and $F^\ell(x) \notin \Delta^\circ(d_1d_2\cdots d_{m-1}e_m)$ for any $\ell = 0, 1, \dots, k_1$. Then there is a $k_2 > k_1$ so that $F^{k_2}(x) \in \Delta^\circ(d_1d_2\cdots d_{m-1}e_m) \subseteq \Delta^\circ(d_1d_2\cdots d_n)$. \square

Proposition 22. *If F satisfies TT then so does the corresponding shift X , and \tilde{X} satisfies TTT.*

Proof. For $w_1 = d_1d_2\cdots d_m, w_2 = e_1e_2\cdots e_k \in \mathcal{L}$,

$$\Delta^\circ(d_1, d_2, \dots, d_{m_1}), \Delta^\circ(e_1e_2\cdots e_{m_2}) \neq \emptyset.$$

Choose $x \in B$ so that $O^+(x)$ is dense. By Lemma 21 there exist $k_2 > k_1 + m_1$ so that $F^{k_1}(x) \in \Delta^\circ(d_1, d_2, \dots, d_{m_1})$ and $F^{k_2}(x) \in \Delta^\circ(e_1e_2\cdots e_{m_2})$. Then $\mathbf{r}(x)_{[k_1, \dots, k_1+m_1-1]} = w_1$ and $\mathbf{r}(x)_{[k_1, \dots, k_2+m_2-1]} = w_2$. Thus $w_1uw_2 \in \mathcal{L}$. \square

Fixing $d \in \mathcal{D}$, let $\overline{\Delta}(d) = [a_d, b_d]$ and let $\alpha_d = \lim_{x \rightarrow a_d^+} F(x)$ and $\beta_d = \lim_{x \rightarrow b_d^-} F(x)$. Define $f_d : [0, 1] \rightarrow [0, 1]$ by

$$(5) \quad f_d(x) = \begin{cases} a_d & \text{if } 0 \leq x < F(\alpha_d) \\ (F|_{\Delta(d)})^{-1}(x) & \text{if } F(\alpha_d) \leq x < F(\beta_d) \\ \beta_d & \text{if } F(\beta_d) \leq x < 1 \end{cases}$$

Each f_d is continuous because $F|_{\Delta(d)} : \Delta(d) \rightarrow [0, 1]$ is continuous and monotonic.

Lemma 23. *If $d_1d_2\cdots d_n \in \mathcal{L}$ then*

$$\overline{\Delta}(d_1d_2\cdots d_n) = f_{d_1}(f_{d_2}(\cdots f_{d_n}([0, 1])\cdots)).$$

Proof. For $n = 1$ we have $f_{d_1}([0, 1]) = [a_1, b_1] = \overline{\Delta}(d_1)$. Suppose

$$f_{d_2}(f_{d_3}(\cdots f_{d_n}([0, 1])\cdots)) = \overline{\Delta}(d_2d_3\cdots d_n) = [a', b'],$$

where $b' > a'$. Note that a' and b' are $f_{d_2}(f_{d_3}(\cdots f_{d_n}(0)\cdots))$ and $f_{d_2}(f_{d_3}(\cdots f_{d_n}(1)\cdots))$ (in one order or the other). Then

$$f_{d_1}(f_{d_2}(\cdots f_{d_n}([0, 1])\cdots)) = f_{d_1}(\overline{\Delta}(d_2d_3\cdots d_n)) = f_{d_1}([a', b']).$$

Now for any interval $[a', b']$, and any $d \in \mathcal{D}$, (5) implies that $f_d([a', b']) = F^{-1}([a', b']) \cap \overline{\Delta}(d)$. The result now follows by (4). \square

Theorem 24. *Let F be a PIM such that F -representations are valid. Then for Lebesgue almost every $x \in [0, 1)$ (i.e., for $x \in B$)*

$$(6) \quad x = E(\mathbf{d}) = \lim_{n \rightarrow \infty} f_{d_0}(f_{d_1}(\dots f_{d_n}(0) \dots)) = \lim_{n \rightarrow \infty} f_{d_0}(f_{d_1}(\dots f_{d_n}(1) \dots)),$$

where $\mathbf{d} = .d_0 d_1 d_2 \dots = \mathbf{r}(x)$.

For $.d_1 d_2 d_3 \dots \in \mathcal{D}^{\mathbb{N}}$ we call the limits of the type (6) *generalized f -expansions*. The conclusion of Theorem 6 can be expressed by saying if F -expansions are valid, then a.e. f -expansion converges to “what it should”. This occurs whenever F satisfies either TT or PTT.

Traditionally, additional assumptions on F allow the limits in (6) to be expressed in a simpler form. These assumptions, which we try and state here fairly generally, involve a more stringent order relations on the digit set \mathcal{D} . We say F *well ordered* if $\mathcal{D} \subseteq \mathbb{Z}$ and $\Delta(d) < \Delta(e)$ if and only if $d < e$ (one may need to relabel \mathcal{D} to make this happen). An example of F that is not well ordered is the Cantor transformation in Example 6. If F is well-ordered, we define $f : \mathbb{R} \rightarrow [0, 1)$ by $f(x) = f_d(x - d)$ if $x \in [d, d + 1)$ for each $d \in \mathcal{D}$. We extend f to a complete the definition of f to a function $f : \mathbb{R} \rightarrow [0, 1]$ by defining $f(x) = f(a)$ for all $x < a$, where $\Delta(d) = [a, b)$ is the left most fundamental interval, and $f(x) = f(b)$ if $[a, b)$ is the first fundamental interval smaller than x . This is most natural if F is either type A or type B, in which case f is continuous, and either increasing or decreasing (not necessarily strictly), respectively.

If we restrict the function f , as defined above, to the intervals in \mathbb{R} on which it is strictly monotonic, then f^{-1} exists, and we have

$$F(x) = f^{-1}(x) \bmod 1.$$

This is a traditional starting point for the theory (see [11], [21]) Equivalently, we can view f as the inverse of the function $F(x) + \xi(x)$.

Given $.d_1 d_2 d_3 \dots \in \mathcal{D}^{\mathbb{N}}$ we define the (classical) f -expansion by

$$f(d_1 + f(d_2 + f(d_3 + \dots))).$$

In particular, we understand this expression this to be the limit

$$\lim_{n \rightarrow \infty} f(d_1 + f(d_2 + f(d_3 + \dots f(d_n) \dots)))$$

Theorem 25. *Suppose F is a well ordered PIM such that F -representations are valid (i.e., if F satisfies either TT or PTT). Then f -expansions are valid in the sense that for λ a.e $x \in [0, 1)$ (i.e., for $x \in B$), $\mathbf{r}(x) = .d_1 d_2 d_3 \dots \in \mathcal{D}^{\mathbb{N}}$ and*

$$x = f(d_1 + f(d_2 + f(d_3 + \dots))).$$

We also have $\lim_{n \rightarrow \infty} f(d_1 + f(d_2 + f(d_3 + \dots f(d_n + 1) \dots)))$

6. TOPOLOGICAL TRANSITIVITY IMPLIES PARRY TOPOLOGICAL TRANSITIVITY

We can now prove our main result.

Theorem 26. *If F is a piecewise interval map (PIM) that that satisfies TT, then it satisfies PTT.*

Proof. Since F satisfies TT, Proposition 22 implies that the 2-sided F -shift \tilde{X} satisfies TTT. Let $\tilde{\mathbf{d}} \in \tilde{X}$ be such that $O^-(\tilde{\mathbf{d}})$ is dense. Let $\mathbf{d}_n = \tilde{T}^{-n}(\tilde{\mathbf{d}})$, $n \geq 0$, and for each n let $\mathbf{d}_n = \pi_+(\tilde{\mathbf{d}}_n)$, where $\pi_+ : \tilde{X} \rightarrow X$, defined $\pi_+(\dots d_{-1} d_0 . d_1 d_2 \dots) =$

$.d_1d_2\dots$, is the factor map from the 2-sided to 1-sided shift. Note that $\pi_+(\tilde{T}(\tilde{\mathbf{d}})) = T(\pi_+(\tilde{\mathbf{d}}))$, so we have

$$T^n(\mathbf{d}_n) = T^n(\pi_+(\tilde{\mathbf{d}}_n)) = \pi_+(\tilde{T}^n(\tilde{\mathbf{d}}_n)) = \pi_+(\tilde{\mathbf{d}}) = \mathbf{d}.$$

Let $x_n = E(\mathbf{d}_n)$, which exists by Theorem 11 and Theorem 24. It follows that $F^n(x_n) = x$ so $B = \{x_0, x_1, x_2, \dots\}$ is a backward orbit for F and it suffices to show B is dense. But $(\tilde{T}^{-n}(\tilde{\mathbf{d}}))|_{[1,2,\dots,m]} = d_1d_2\dots d_m$ implies $x_n \in \overline{\Delta}(d_1d_2\dots d_m)$. Since $O^-(\tilde{\mathbf{d}})$ is dense, B is dense too, and so F satisfies PTT. \square

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DEPARTMENT OF MATHEMATICS, GEORGE WASHINGTON UNIVERSITY, 2115 G ST. NW, WASHINGTON, DC 20052

E-mail address: `robinson@gwu.edu`